

The geometrodynamical origin of equilibrium gravitational configurations

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The origin of equilibrium gravitational configurations is sought in terms of the stability of their trajectories, as described by the curvature of their Lagrangian configuration manifold. We focus on the case of spherical systems, which are integrable in the collisionless (mean field) limit despite the apparent persistence of local instability of trajectories even as $N \rightarrow \infty$. It is shown that when the singularity in the potential is removed, a null scalar curvature is associated with an effective, averaged, equation of state describing dynamically relaxed equilibria with marginally stable trajectories. The associated configurations are quite similar to those of observed elliptical galaxies and simulated cosmological halos. This is the case because a system starting far from equilibrium finally settles in a state which is integrable when unperturbed, but where it can most efficiently wash out perturbations. We explicitly test this interpretation by means of direct simulations.

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INTRODUCTION

In a contemporary context, the Newtonian dynamics of a large number of self-gravitating particles describes the dynamics of putative dark matter, and to first order the formation and evolution of galaxies and clusters. Galaxies that are thought to have formed through largely dissipationless processes show remarkable regularity in their density profiles, and dark matter halos identified in cosmological simulations have similar (and nearly) universal profiles [1, 2]. Yet there is no satisfactory theory predicting these preferred choices of dynamical equilibria.

Steady states are characterized by the absence of coherent modes, which decay due to differences between motions on neighboring trajectories. This effect is present in all nonlinear systems (phase mixing [3]), but is most efficient if nearby trajectories rapidly diverge and spread in phase space. However, attempts at relating these trajectory stability properties to the choice of N -body equilibrium — or extracting almost any interesting information at all from them — are frustrated by a puzzling phenomenon. As $N \rightarrow \infty$, particles move in the mean field potential, and steady state systems with sufficient symmetry become near integrable — e.g., spherical systems are completely integrable when in equilibrium, and phase space distances between their trajectories thus generally diverge linearly in time [3]. In contrast, it has long been realized that direct integrations of the equations of motion invariably reveal exponential divergences on a timescale that does not increase with N [4, 5].

A clue concerning the resolution of this paradox comes from studies of softened systems. For when the singularity in the potential is removed (e.g. $\phi \sim 1/r \rightarrow \phi \sim 1/\sqrt{r^2 + \text{const}}$), the divergence timescale does increase with N [6, 7]. The purpose of the present study is to show that, in this case, physically interesting information, pertaining to the choice of equilibrium, can be extracted from the stability properties of trajectories.

EQUILIBRIUM AND PRESSURE SUPPORT

The divergence of nearby trajectories can also be deduced from geometric representations of motion and stability in gravitational systems (e.g., [8–11]). Though the geometric description is not unique [12], one well known formulation has the advantage of involving a positive definite metric defined directly on the Lagrangian configuration manifold — a curved subspace of the $3N$ -dimensional space spanned by Cartesian particle coordinates, the phase space being its cotangent bundle [13]. This conformally flat metric is expressed in terms of the Cartesian coordinates and the system kinetic energy, $W = E - V$, by $ds^2 = W \sum_{3N} (dx^\alpha)^2$ (e.g., [13, 14]). The divergence of trajectories is then determined by the two dimensional sectional curvatures $k_{\mathbf{u}, \mathbf{n}}$, where \mathbf{u} is the system velocity vector and \mathbf{n} define directions normal to it. When the $k_{\mathbf{u}, \mathbf{n}}$ are negative, trajectories are exponentially unstable [15]. Isotropic N -body systems with singular two-body potentials are predicted to tend toward this state as N is increased and direct two-body collisions are ignored [8, 9]; and trends similar to those obtained from direct calculations are recovered when the full gravitational field is taken into account [9].

We will be interested in how N -body systems damp out fluctuations, we therefore seek an averaged measure of their response to random perturbations. Integrating over directions \mathbf{n} , we obtain the mean (or Ricci) curvature $r_{\mathbf{u}} = \sum_{\mu=1}^{3N-1} k_{\mathbf{n}_\mu, \mathbf{u}}(s)$. Furthermore, we will focus on isotropic systems. In this case, a further average over the velocity directions \mathbf{u} can be made, giving the Ricci scalar; which, for $N \gg 1$, can be expressed as

$$R = \sum_{\mathbf{u}, \mathbf{n}} k_{\mathbf{u}, \mathbf{n}} = -3N \frac{\nabla^2 W}{W^2} - 9N^2 \frac{\|\nabla W\|^2}{4W^3}, \quad (1)$$

where ∇W and $\nabla^2 W$ are the gradient and Laplacian, taken with respect to coordinates $q_i = \sqrt{m_p x_i}$, and m_p are the particle masses (which we will assume to be

equal). For sufficiently large- N spherical systems with isotropic velocities $r_u \rightarrow R/N$ [16].

A correspondingly averaged equation, describing stability in the time domain, is given by (e.g., [11])

$$\frac{d^2 Y}{dt^2} + Q(t)Y = 0. \quad (2)$$

Here Y measures the mean divergence of trajectories and

$$Q = \frac{2W^2}{9N^2} R - \frac{1}{4} \left(\frac{\dot{W}}{W} \right)^2 + \frac{1}{2} \frac{d}{dt} \left(\frac{\dot{W}}{W} \right). \quad (3)$$

Near dynamical equilibrium, $\dot{W} \rightarrow 0$ and the dominant term on the right hand side will be the curvature term. For singular Newtonian potentials, and when direct collisions are ignored, Eq. (1) predicts negative R (since the first term vanishes), and exponential instability is present even in the case of spherical equilibria. Omitting direct collisions however implies an incomplete manifold; it is not clear if straightforward application of the geometric approach remains appropriate in this case [17].

Formally, as $N \rightarrow \infty$, the first term in Eq. (1) would contribute through a collection of delta functions; but if the singularity in the potential is softened, this term represent a smoothed out density field integrated over the particle distribution (with the softening length acting as smoothing parameter), and, in dynamical equilibrium, the two terms on the right hand side of Eq. (1) become comparable. Indeed, by multiplying the terms by W^2/N^2 , taking the inverse, and the square root, these terms can be seen to be linked to the natural timescale of the system (the dynamical, or crossing, time) — the first through the relation $\tau_D \sim 1/(G\bar{\rho})^{1/2}$ (where the average density is related to the Laplacian via the Poisson equation), and the second through $\tau_D \sim (W/m_p)^{1/2} \bar{a}^{-1}$, where \bar{a} is the average acceleration affecting a particle.

In fact, the structure of Eq. (1) reflects a pressure-gravity balance. To see this, assume $R = 0$. Then

$$4\pi G W \sum_i \rho_i = \frac{3}{4} N m_p \sum_i a_i^2, \quad (4)$$

where ρ_i and a_i refer to the density and acceleration at the position of particle i . This equates a density multiplied by the velocities squared (i.e., a pressure term) to a gravitational binding term. In the continuum approximation, and for a spherical system, this becomes

$$4\pi W \int \rho^2 r^2 dr = \frac{3}{4} M G \int \frac{m^2}{r^2} \rho dr, \quad (5)$$

where $\rho = \rho(r)$ is the radial density, m is the mass enclosed within radius r , M is the total mass, and the integrals are evaluated over the volume of the configuration. In dynamical equilibrium one can use the Jeans equation for an isotropic system (e.g., [3]) to get

$$4\pi W \int \rho^2 r^2 dr = -\frac{3}{4} M \int m d(\rho \sigma^2), \quad (6)$$

where $\sigma^2 = \bar{v}^2(r)$ is the one-dimensional velocity dispersion. There is one easily identifiable solution of Eq. (6); it corresponds to $\rho \sim 1/r^2$ and $\sigma = \text{constant}$. This is a singular isothermal sphere; though its gravitational force, and both terms in Eq. (6), diverge as $r \rightarrow 0$, and the mass diverges as $r \rightarrow \infty$, there is significance to this solution, as isothermal spheres enclosed by a boundary are entropy maxima of softened gravitational systems, which do tend toward these maxima, provided certain conditions are met [18, 19].

For well behaved systems, $\rho \sigma^2 = 0$ as $r \rightarrow 0$ and $r \rightarrow \infty$. Integrating by parts, noting this, and that the kinetic energy is related to the average velocity dispersion by $W = \frac{3}{2} M \langle \sigma^2 \rangle_r$, and $dm = 4\pi r^2 \rho dr$, one gets

$$\langle \sigma^2 \rangle_r \int \rho dm = \frac{1}{2} \int \rho \sigma^2 dm. \quad (7)$$

If well defined pressure and temperature functions can be assigned (i.e., assuming local thermodynamic equilibrium), then $P = \rho \sigma^2$ and $W = \frac{3}{2} N k \langle T \rangle_r$, and

$$\int P dm = 2 \frac{k \langle T \rangle_r}{m_p} \int \rho dm. \quad (8)$$

Save for the integrations over the mass distribution, this form is that of an ideal gas equation of state. Nevertheless, because of the factor 2, there are no associated isothermal states; in contrast, solutions require that the density and velocity dispersion distributions correlate.

APPROACH TO EQUILIBRIUM: AN INTERPRETATION

The results obtained thus far suggest that dynamically relaxed equilibrium systems should have small or vanishing R ; but how do they reach such configurations?

The density and potential of a system started far from equilibrium will fluctuate until collective motions are damped; this will take place most efficiently when it finds a configuration where its response is least coherent; that is, when the divergence of neighboring trajectories are maximal, and coherent modes are damped through phase space mixing of trajectories as the system responds to density and potential fluctuations. At the same time, it is required that in the absence of fluctuations the trajectories of a spherical system should not be unstable, so as to recover an integrable system in the infinite- N limit. This implies that R cannot be negative at equilibrium.

Eq. (2) suggests that configurations that satisfy these requirements have $R \rightarrow 0$ at equilibrium: a system starting far from equilibrium should then end up in a configuration satisfying this condition. To show this quantitatively, we have used a formulation, developed by Pettini and collaborators, which assumes that, for positive Q , unstable solutions of Eq. (2) can be estimated by dividing Q into a mean term k_0 and rms fluctuations σ_k with

characteristic timescale τ ; this gives rise to an effective Lyapunov exponent (e.g., Eqs. 79 of [12])

$$\lambda = \frac{1}{2} \left(\Lambda - \frac{4k_0}{3\Lambda} \right), \quad (9)$$

with $\Lambda^3 = \sigma_k^2 \tau + \sqrt{\left(\frac{4k_0}{3}\right)^3 + \sigma_k^4 \tau^2}$.

In general, phase space averages are not well defined for gravitational systems (which have a non-compact phase space). Nevertheless, every exact ($N \rightarrow \infty$) collisionless equilibrium has a characteristic R and associated $k_0 = \frac{2W^2 R}{9N^2}$. It is therefore possible to quantify the effect of fluctuations of a certain magnitude $\sigma_k^2 = \langle Q^2 \rangle - \langle Q \rangle^2$ about a given equilibrium, and to compare the magnitude of this effect for different collisionless configurations.

The natural timescale for fluctuations in a gravitational system is the dynamical (or crossing) time, so $\tau = \tau_D$. And since Q comes in units of inverse time squared, we can write $k_0 = a/\tau_D^2$ and $\sigma = b/\tau_D^2$, and express the exponentiation time associated with trajectory divergence in terms of the natural timescale as

$$\frac{\tau_e}{\tau_D} = \frac{6 \left(b^2 + \sqrt{\left(\frac{4}{3}a\right)^3 + b^4} \right)^{1/3}}{3 \left(b^2 + \sqrt{\left(\frac{4}{3}a\right)^3 + b^4} \right)^{2/3} - 4a}. \quad (10)$$

This is proportional to ab^{-2} for $a \gg b$ and tends to $2^{2/3}b^{-2/3}$ as $b \gg a$. For a given fluctuation level b , the decoherence of trajectories due to local divergence will be more efficient, i.e. its exponential time will be smaller, for smaller a , and therefore k_0 and average R .

TESTING THE INTERPRETATION

We now examine the above interpretation by means of direct N -body simulations. Since we are interested in estimating the densities and accelerations in the continuum limit, we use a technique [20], which expands the density and potential in smooth functional series; and since we are interested in strictly spherical configurations, we only make use of the radial expansions in this scheme (which is carried up to order 30). Systems are sampled using a 100000 particles, started from homogeneous spatial initial conditions inside a unit sphere of unit mass. This configuration has a natural timescale $\tau_D = (G\rho(0))^{-1/2} = \sqrt{4\pi/3}$, which will be our time unit.

We have run configurations starting with isotropic velocities that are either constant or following various decreasing or increasing functions of radius. The results shown here correspond to the latter case, for the following reason. Collisionless evolution implies that the 'pseudo-phase space density' $\rho_p = \rho/\sigma^3$ generally decreases along particle trajectories [3]; and one of our aims is to explain the dynamical origin of cosmological halos, which have (a

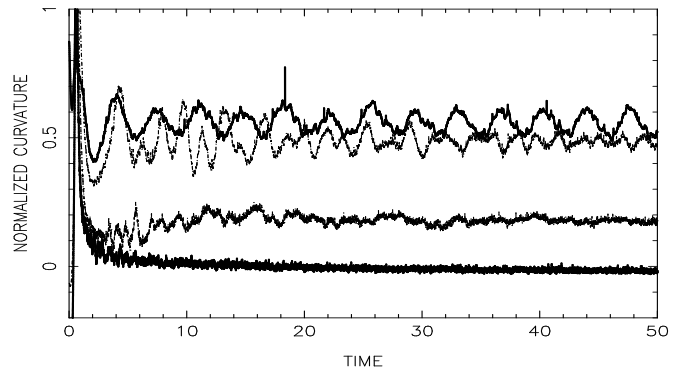


FIG. 1. Time evolution of normalized curvature R_n (Eq. 11), measured in initial dynamical times, for systems starting with (from top to bottom) virial ratios, 1, 0.5, 0.25 and 0.125

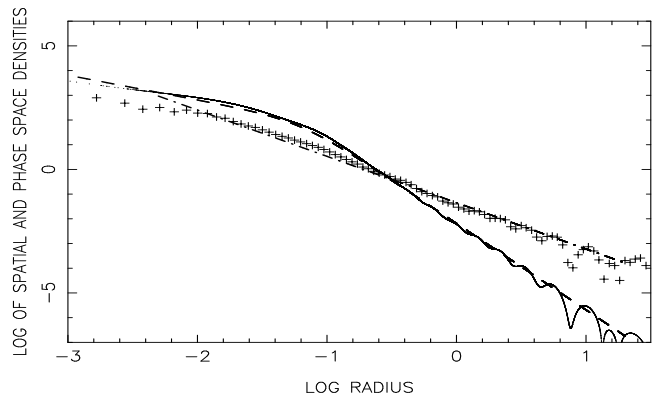


FIG. 2. Equilibrium density (solid line) and phase space density (crosses) of system starting with virial ratio 0.125. Dashed lines are eye fits with $\rho \sim 1/(r(1+r^{2.5}))$ and $\rho_p \sim r^{1.875}$

centrally divergent) $\rho_p \sim r^{-1.875}$ [21]. From among various functional forms tried, results will be shown for initial $\sigma \sim r$, noting that the trends reported seem generic.

We vary the initial virial ratio $\text{Vir} = 2W/|V|$ from 1, corresponding to near equilibrium, to 0.125, corresponding to a system started far from equilibrium. Given the sampling errors for finite N , R will never tend to zero exactly in practice; and the absolute values of the densities and accelerations (and local dynamical times) strongly depend on the central concentrations of the final configurations. A relative measure is thus required for comparing these. We use the normalized quantity

$$R_n = \left(4\pi G W \sum_1^N \rho_i \right) \left(\frac{3}{4} N m_p \sum_1^N a_i^2 \right)^{-1} - 1, \quad (11)$$

which measures the departure from equality in Eq. (4).

The results are in Fig. 1, which shows that the equilibrium R_n is indeed always of order unity or less, and that the further from equilibrium the initial state, the closer the evolved system is to a configuration that minimizes R . The initial R_n is zero when $\text{Vir} = 0.5$; systems with initial $\text{Vir} < 0.5$ start with $R < 0$ and then explore state

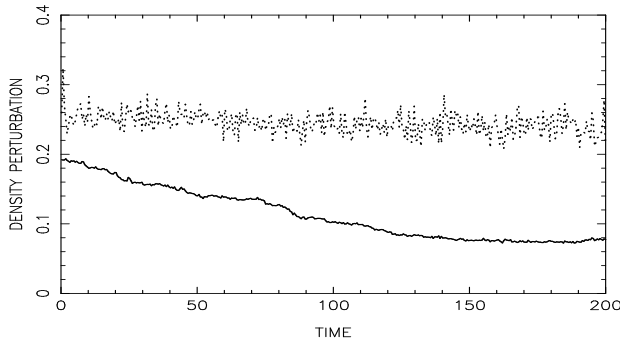


FIG. 3. Evolution of density perturbations, defined as $\sum_{i=1}^N (\rho_i(t=0) - \rho_i(t))^2 / \sum_{i=1}^N \rho_i^2(t=0)$, when coordinates of final states of systems starting from virial ratio 0.125 (solid line) and 1 (dotted line) are perturbed by 10%

space until they achieve configurations where (according to Eq. 10) oscillations are efficiently damped.

Also, as shown in Fig. 2, the equilibrium density and phase space density profiles of this system are remarkably similar to those of cosmological halos (and shallow cusp elliptical galaxies [2]). There are some differences. Our outer density profile is steeper than the $1/r^3$ form of cosmological halos; however finite configurations cannot have $1/r^3$ outer profiles. (Cosmological halos are not isolated equilibrium structures as their outer regions are subject to mass infall.) The functional form of our empirical spatial density fit is also slightly different from standard 'NFW' (where $\rho_{\text{NFW}} \sim \frac{1}{r(1+r)^2}$), and there seems to be some flattening toward the center of the $1/r$ cusp. Yet, similar departures from ideal NFW often also apply to cosmological halos [22].

Finally, we explicitly verify that configurations with smaller R_n do efficiently damp out fluctuations. This is already apparent in Fig. 1, where fluctuations are quickly damped for such systems. (This may be intuitively expected, for these systems lack homogeneous harmonic cores with nearly constant orbital frequencies and coherent response.) Further verification is obtained by perturbing the final states and measuring variations in the density distributions as systems, so perturbed, are evolved. The results of one such experiment, where the spatial coordinates were decreased by 10%, are shown in Fig. 3. There, the system that was perturbed from a state with smaller R_n is seen to quickly settle back to an equilibrium that is closer to the unperturbed one, and to gradually close in further on it (as the outer regions, with large crossing times, evolve).

CONCLUSION

Contradictions between predictions pertaining to the dynamical stability of large- N body system trajectories and those implied by the collisionless limit disappear

when the singularity in the potential is removed. What transpires instead is a description of gravitational equilibria. We propose an interpretation of the approach to equilibrium that is effectively a rationalization of the classic idea of 'violent relaxation' [23]: systems starting far from equilibrium reach such a state once a configuration where collective oscillations are efficiently damped is reached, and such configurations correspond to states whose dynamical trajectories are marginally stable. Imposing this condition leads to relations describing relaxed dissipationless equilibrium configurations that are remarkably similar to those observed and modeled. These systems do not 'ring' and are more robust.

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